

AD-A141 700

ON A CLASS OF QUASILINEAR PARTIAL INTEGRODIFFERENTIAL  
EQUATIONS WITH SING. (U) WISCONSIN UNIV-MADISON

1/1

MATHEMATICS RESEARCH CENTER W J HRUSA ET AL. APR 84  
MRC-TSR-2667 DAAG29-80-C-0041

UNCLASSIFIED

F/G 12/1

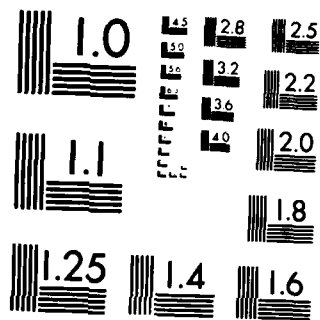
NL

END

FORM

7 84

DTIC



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A141 700

MRC TECHNICAL SUMMARY REPORT # 2667

ON A CLASS OF QUASILINEAR  
PARTIAL INTEGRODIFFERENTIAL  
EQUATIONS WITH SINGULAR KERNELS

W. J. Hrusa and M. Renardy

**Mathematics Research Center  
University of Wisconsin—Madison  
610 Walnut Street  
Madison, Wisconsin 53705**

April 1984

(Received February 6, 1984)

DTIC FILE COPY

Approved for public release  
Distribution unlimited

DTIC  
EXTRACTED

JUN 01 1984

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

National Science Foundation  
Washington, DC 20550

84 05 31 067

UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

ON A CLASS OF QUASILINEAR PARTIAL INTEGRODIFFERENTIAL  
EQUATIONS WITH SINGULAR KERNELS

W. J. Hrusa<sup>1,2</sup> and M. Renardy<sup>1,3</sup>

Technical Summary Report #2667

April 1984

DEDICATED TO JOHN NOHEL ON THE OCCASION OF HIS  
SIXTIETH BIRTHDAY

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail. and/or Special
A-1	



ABSTRACT

We prove local and global existence theorems for a model equation in nonlinear viscoelasticity. In contrast to previous studies, we allow the memory function to have a singularity. We approximate the equation by equations with regular kernels and use energy estimates to prove convergence of the approximate solutions.

AMS (MOS) Subject Classifications: 35L15, 45E10, 45K05, 73F15, 76A10

Key Words: Nonlinear viscoelasticity, hyperbolic equations, singular integrals, Volterra equations, energy estimates.

Work Unit Number 1 - Applied Analysis

---

<sup>1</sup> Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

<sup>2</sup> This material is based upon work supported by the National Science Foundation under Grant No. MCS-8210950.

<sup>3</sup> This material is based upon work supported by the National Science Foundation under Grant Nos. MCS-8210950 and MCS-8215064.

## SIGNIFICANCE AND EXPLANATION

In a recent paper, Dafermos and Nohel considered a model equation for nonlinear viscoelasticity. They proved that smooth solutions exist locally in time and also globally in time for small data. For large data, globally defined smooth solutions will not exist in general, and formation of shocks is expected.

In the analysis of Dafermos and Nohel, and in other papers showing related results, it is essential that the viscoelastic memory function is absolutely continuous. There are, however, some indications, on both a theoretical and an experimental basis, that certain viscoelastic materials may be adequately described by models with singular memory functions. The mathematical existence properties for such models should in fact be better than for regular memory functions, since a singular memory function precludes the formation of shocks. However, the methods used in previous existence proofs cannot be generalized to singular kernels.

In this paper, <sup>its authors</sup> we provide an existence theory for such models. We approximate the equation by equations with regular kernels, for which existence is known. <sup>They</sup> We then use energy estimates to show that these approximate solutions converge to a limit.

---

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON A CLASS OF QUASILINEAR PARTIAL INTEGRODIFFERENTIAL  
EQUATIONS WITH SINGULAR KERNELS

W. J. Hrusa<sup>1,2</sup> and M. Renardy<sup>1,3</sup>

1. Introduction

Many model equations for viscoelastic materials have the form of a quasilinear hyperbolic equation perturbed by a dissipative integral term of Volterra type. In the recent literature, a number of existence results for such models have been proved [2-4], [10-14], [17], [21-22], [26]. These papers establish the existence of classical solutions to the initial value problem locally in time, and (in some cases) globally in time if the data of the problem are small. For large data, global existence does not hold in general, and shocks are expected to develop [7], [9], [18-19], [25].

Common to all the works referred to above is the assumption that the kernel in the integral term has sufficient regularity. We are here interested in the possibility that this kernel is singular at 0. Kinetic theories for chain molecules [5], [24], [28] and some experimental data [15] suggest that this is a realistic possibility, at least for some viscoelastic materials. Although some rheological properties of models with singular kernels have been investigated (see e.g. [1]), there do not seem to be many studies from a fundamental mathematical point of view.

The only existence theorem for models with singular kernels that we are aware of is a result by Londen [16] concerning the existence of weak solutions. His class of equations includes the case  $\psi \equiv \phi$  in the problem introduced below. Londen's assumptions require the viscoelastic memory function to have a singularity which is stronger than logarithmic.

Renardy [23] has studied linear wave propagation. His results show that certain singular kernels do not permit propagation of singularities and have a smoothing effect.

---

<sup>1</sup>

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

<sup>2</sup>

This material is based upon work supported by the National Science Foundation under Grant No. MCS-8210950.

<sup>3</sup>

This material is based upon work supported by the National Science Foundation under Grant Nos. MCS-8210950 and MCS-8215064.

Hannsgen and Wheeler [8] show (for the constant coefficient linear problem on a bounded domain) that the evolution operator is compact for positive time if and only if the kernel is singular. This suggests that, if anything, these models should have "nicer" existence properties than those with regular kernels. However, this also indicates that one cannot expect the methods of previous existence proofs to extend to singular kernels. These proofs rely on an iteration scheme that treats the hyperbolic part as the principal term and the integral as a perturbation. This, of course, works irrespective of the sign of the integral. If, however, singular kernels lead to smoothing, then the reverse sign of the integral must lead to blow-up, and a local existence theorem cannot hold.

In this paper, we focus on the history value problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_{-\infty}^t a'(t-\tau)\psi(u_x(x,\tau))_x d\tau + f(x,t), \quad (1.1)$$

$$0 < x < 1, \quad -\infty < t < \infty,$$

$$u(0,t) = u(1,t) = 0, \quad -\infty < t < \infty, \quad (1.2)$$

$$u(x,t) = v(x,t), \quad 0 < x < 1, \quad -\infty < t < 0, \quad (1.3)$$

which was studied by Dafermos and Nohel [4]. (Closely related problems with regular kernels have also been studied by MacCamy [17], Dafermos and Nohel [3], Staffans [26], Hattori [9], and Hrusa and Nohel [13]. See [12] for a summary of these works.)

Like Dafermos and Nohel, we assume  $\phi(0) = \psi(0) = 0$ ,  $\phi' > 0$ ,  $\psi' > 0$ ,  $\phi' - a(0)\psi' > 0$ . They require that the kernel  $a$  is strongly positive definite; for technical reasons we make the stronger assumption that  $a$  is positive, monotone decreasing, and convex. While they assume that  $a, a', a'' \in L^1(0, \infty)$ , we allow  $a'$  to have a singularity at 0, e.g.  $a'(t) \sim -t^{-\alpha}$ ,  $0 < \alpha < 1$ , as  $t \rightarrow 0$ .

For definiteness, we shall always consider (1.1) with Dirichlet boundary conditions (1.2). We emphasize, however, that our local existence proof can be applied without change for Neumann or mixed boundary conditions or for the all-space problem. We have purposely avoided the use of Poincaré inequalities in our estimates for this reason. The global result can also be generalized to different boundary conditions. For the case of Neumann conditions, we need a trivial modification in the statement of the theorem, due to the possibility of rigid motions which need not decay as  $t \rightarrow \infty$ . We do not know how to extend

the global result to the all-space problem. Recent work on this problem by Hrusa and Nohel [13] makes very essential use of the assumption that the kernel is regular.

It is not easy to quantify the regularizing effect of a singular kernel in general terms. Roughly speaking, certain types of waves are smoothed, while others are not. For those waves that are smoothed, the precise degree of smoothing depends crucially on the nature of the singularity in the kernel. This will be discussed in detail for linear problems in a future work.

In our treatment, we regard (1.1) as a history value problem with a history which is assumed to satisfy the equation and boundary conditions, and a "smooth" forcing term. This ensures that we satisfy compatibility conditions between the initial and boundary conditions as well as compatibility conditions between derivatives of the history and derivatives of the solution for  $t > 0$ . It is possible to relax the assumption that the history satisfies the equation, with the result that derivatives of  $u$  may be discontinuous across  $t = 0$ . Formally, we could also replace the history value problem by an initial value problem if we set  $\int_{-\infty}^t = \int_{-\infty}^0 + \int_0^t$  and incorporate the first part in the forcing term. This leads to a technical inconvenience making the statement of results rather complicated. If  $u$  is a smooth function and the kernel is singular, then the integral in (1.1) is also a smooth function, but the separate integrals  $\int_{-\infty}^0$  and  $\int_0^t$  have cancelling singularities at  $t = 0$ . Thus, if formulated for the initial value problem, our results would involve a singular forcing term. For an initial value problem with a singular kernel and smooth forcing term, the solution  $u$  will have a singularity in the time direction as  $t \rightarrow 0$ .

The paper is organized as follows. In Section 2, we prove some preliminary lemmas concerning the kernel. In Section 3 we prove an existence result for linear problems with variable coefficients. This is done by approximating the problem by problems with regular kernels, for which existence is known. We then use energy estimates that hold uniformly as the kernel becomes singular to show that the solutions of these approximate problems converge to a limit. In Section 4, we establish local existence for the nonlinear problem



by using the results of Section 3 and a contraction argument. Section 5 contains a brief discussion of global existence. We notice that once local existence is known, the assumption  $a'' \in L^1$  is not essential for the global existence proof of Dafermos and Nohel and can be avoided by a minor modification.

Our global existence theorem requires the data to be small. It is conceivable that for certain singular kernels, global smooth solutions of (1.1), (1.2), (1.3) also exist for large data. However, we have been unable to verify this.

With the exception of Section 2, subscripts  $x$  and  $t$  indicate partial differentiation. A prime denotes the derivative of a function of a single variable, and we use the symbol  $:=$  for an equality in which the left hand side is defined by the right hand side. All derivatives should be interpreted in the distributional sense.

Acknowledgement: We thank J. M. Wilson for a helpful discussion.

## 2. Preliminaries

This section contains some preliminary results (concerning the kernel  $a$ ) that will be used in the subsequent sections. Let  $H$  be a complex Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and associated norm  $\|\cdot\|$ . For each  $b \in L^1(0, \infty)$ ,  $T \in \mathbb{R}$ , and  $u \in L^2((-\infty, T]; H)$ , we set

$$Q(u, t, b) := \int_{-\infty}^t \langle u(s), \int_{-\infty}^s b(s-\tau) u(\tau) d\tau \rangle ds \quad \forall t \in (-\infty, T] \quad (2.1)$$

We use a hat to denote the Laplace transform evaluated along the imaginary axis, i.e.

$$\hat{q}(\omega) := \int_0^\infty e^{-i\omega t} q(t) dt \quad \forall \omega \in \mathbb{R}, \quad (2.2)$$

for real and  $H$ -valued functions  $q$ . For  $T \in \mathbb{R}$ ,  $h > 0$ ,  $u: (-\infty, T] \rightarrow H$ , and  $t \in (-\infty, T]$ , we employ the notations

$$\Delta_h u(t) := u(t) - u(t-h) \quad \forall t \in (-\infty, T], \quad (2.3)$$

and

$$u_t(\tau) := u(t-\tau) \quad \forall \tau > 0, \quad (2.4)$$

in particular,

$$\hat{u}_t(\omega) := \int_0^\infty e^{-i\omega \tau} u(t-\tau) d\tau \quad \forall \omega \in \mathbb{R}. \quad (2.5)$$

The concept of a strongly positive definite kernel will play a central role in our analysis. We recall that a real-valued function  $b \in L^1_{loc}(0, \infty)$  is said to be positive definite (or of positive type) if

$$\int_0^t w(s) \int_0^s b(s-\tau) w(\tau) d\tau ds > 0 \quad \forall t > 0, \quad (2.6)$$

for every  $w \in C(0, \infty)$ ;  $b$  is called strongly positive definite if there exists a constant  $\lambda > 0$  such that the function defined by  $b(t) - \lambda e^{-t}$ ,  $t > 0$ , is positive definite. As the terminology suggests, strongly positive definite implies positive definite.

Throughout this section, we assume that

$$a, a' \in L^1(0, \infty), \quad a \text{ is strongly positive definite}. \quad (2.7)$$

It follows from (2.7) that  $a \in AC(0, \infty)$ ,  $a(0) > 0$ , and\*

$$\operatorname{Re} \hat{a}(\omega) > \frac{\lambda}{\omega^2 + 1} \quad \forall \omega \in \mathbb{R}, \quad (2.8)$$

---

\* In fact, for  $a \in L^1(0, \infty)$  to be strongly positive definite it is necessary and sufficient that (2.8) hold for some  $\lambda > 0$ .

for some constant  $\lambda > 0$ . Consequently,  $\text{Re } \hat{a}$  is integrable and

$\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re } \hat{a}(\omega) d\omega = \frac{1}{2} a(0)$ . (See, for example, [20] for more information on strongly positive definite kernels.)

In our analysis of equation (1.1), terms of the form  $\lim_{h \rightarrow 0} \frac{1}{h^2} Q(\Delta_h u, t, a)$  will arise, where it is known a priori merely that  $u \in L^2((-\infty, T]; H)$ . Of course, this is not sufficient to guarantee that the limit in question exists. However, if we know from other considerations that the limit does exist, some rather useful conclusions can be drawn.

**Lemma 2.1:** Let  $T \in \mathbb{R}$  and  $u \in L^2((-\infty, T]; H)$  be given. Assume that (2.7) holds and that  $\lim_{h \rightarrow 0} \frac{1}{h^2} Q(\Delta_h u, t, a)$  exists for a.e.  $t \in (-\infty, T]$ . Then, for a.e.  $t \in (-\infty, T]$ ,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h^2} Q(\Delta_h u, t, a) &= \frac{1}{2} a(0) \|u(t)\|^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 \text{Re } \hat{a}(\omega) \|\hat{u}_t(\omega)\|^2 d\omega \\ &\quad - \frac{1}{\pi} \langle u(t), \int_{-\infty}^{\infty} (\text{Im } \hat{a}(\omega)) \hat{u}_t(\omega) d\omega \rangle. \end{aligned} \quad (2.9)$$

In particular, each term in (2.9) is well-defined for a.e.  $t \in (-\infty, T]$ .

**Proof:** For each  $h > 0$ , we have

$$\frac{1}{h^2} Q(\Delta_h u, t, a) = \frac{1}{2\pi h^2} \int_{-\infty}^{\infty} \text{Re } \hat{a}(\omega) \|\hat{u}_t(\omega) - \hat{u}_{t-h}(\omega)\|^2 d\omega, \quad (2.10)$$

by Parseval's identity. Next, we observe that

$$\begin{aligned} \hat{u}_{t-h}(\omega) &= \int_0^{\infty} u(t-h-\tau) e^{-i\omega\tau} d\tau \\ &= \int_h^{\infty} u(t-\sigma) e^{-i\omega\sigma} e^{i\omega h} d\sigma \\ &= e^{i\omega h} \hat{u}_t(\omega) - e^{i\omega h} \int_0^h u(t-\sigma) e^{-i\omega\sigma} d\sigma, \end{aligned} \quad (2.11)$$

and consequently

$$\begin{aligned} \frac{1}{h^2} Q(\Delta_h u, t, a) &= \\ &= \frac{1}{2\pi h^2} \int_{-\infty}^{\infty} \text{Re } \hat{a}(\omega) \|(1 - e^{i\omega h}) \hat{u}_t(\omega) + e^{i\omega h} \int_0^h u(t-\sigma) e^{-i\omega\sigma} d\sigma\|^2 d\omega. \end{aligned} \quad (2.12)$$

Using the fundamental theorem of calculus and the dominated convergence theorem, we see

that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{2\pi h^2} \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega) \|e^{i\omega h} \int_0^h u(t-\sigma) e^{-i\omega \sigma} d\sigma\|^2 d\omega \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega) \|u(t)\|^2 d\omega \\ = \frac{1}{2} a(0) \|u(t)\|^2. \end{aligned} \quad (2.13)$$

(In particular, the limit on the left hand side of (2.13) exists for a.e.  $t \in (-\infty, T]$ .)

The lemma is now immediate if we observe the simple facts that  $\lim_{h \rightarrow 0} \frac{1}{h} (1 - e^{i\omega h}) = -i\omega$ ,  $|\frac{1}{h} (1 - e^{i\omega h})| < |\omega| \quad \forall h > 0$ , and  $\operatorname{Im} \hat{a}(\omega) = \omega \operatorname{Re} \hat{a}(\omega)$ .

It is important to note that the first and second terms on the right hand side of (2.9) are nonnegative. The next lemma provides a useful estimate for the last term in this expression.

**Lemma 2.2:** Assume that (2.7) holds and let  $\varepsilon > 0$  be given. Then, there exists a constant  $C(\varepsilon)$  such that

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} (\operatorname{Im} \hat{a}(\omega)) \hat{u}_t(\omega) d\omega \right\|^2 < \varepsilon \int_{-\infty}^{\infty} \omega^2 \operatorname{Re} \hat{a}(\omega) \|\hat{u}_t(\omega)\|^2 d\omega \\ + C(\varepsilon) \int_{-\infty}^{\infty} \|\hat{u}_t(\omega)\|^2 d\omega \quad \text{a.e. } t \in (-\infty, T], \end{aligned} \quad (2.14)$$

for every  $T \in \mathbb{R}$  and every  $u \in L^2((-\infty, T]; H)$ . (No claim is made that the integrals in (2.14) are all finite.)

**Proof:** Observe that

$$|\operatorname{Im} \hat{a}(\omega)| = \sqrt{|\omega \operatorname{Im} \hat{a}(\omega)|} \cdot \sqrt{\left| \frac{\operatorname{Im} \hat{a}(\omega)}{\omega} \right|}, \quad \omega \neq 0. \quad (2.15)$$

Using (2.15) and the Cauchy-Schwarz inequality, we find that for each  $\alpha > 0$ ,

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} (\operatorname{Im} \hat{a}(\omega)) \hat{u}_t(\omega) d\omega \right\|^2 < 4\alpha \int_{-\alpha}^{\alpha} \|\hat{u}_t(\omega)\|^2 d\omega \cdot \sup_{[-\alpha, \alpha]} |\operatorname{Im} \hat{a}(\omega)|^2 \\ + 2 \left( \int_{\alpha}^{\infty} |\omega \operatorname{Im} \hat{a}(\omega)| \cdot \|\hat{u}_t(\omega)\|^2 d\omega \right) \cdot \left( \int_{\alpha}^{\infty} \left| \frac{\operatorname{Im} \hat{a}(\omega)}{\omega} \right| d\omega \right), \end{aligned} \quad (2.16)$$

where  $A_\alpha := (-\infty, \alpha] \cup [\alpha, \infty)$ . Recalling that  $\text{Im } \hat{a}'(\omega) = \omega \text{Re } \hat{a}(\omega)$  and  $\text{Re } \hat{a}$  is integrable over  $(-\infty, \infty)$ , the lemma follows from (2.16) for a sufficiently large choice of  $\alpha$ . ■

Combining Lemmas 2.1 and 2.2, and making use of the simple algebraic inequality  $|AB| \leq \eta A^2 + \frac{B^2}{4\eta}$  for all  $\eta > 0$ , we easily establish

**Lemma 2.3:** Assume that (2.7) holds. Then, for each  $\varepsilon > 0$  there exists a constant  $C(\varepsilon)$  such that

$$\lim_{h \rightarrow 0} \frac{1}{2} Q(\Delta_h u, t, a) > \left(\frac{1}{2} a(0) - \varepsilon\right) \|u(t)\|^2 - C(\varepsilon) \int_{-\infty}^t \|u(s)\|^2 ds \quad \text{a.e. } t \in (-\infty, T] \quad (2.17)$$

for every  $T \in \mathbb{R}$  and every  $u \in L^2((-\infty, T], H)$  for which  $\lim_{h \rightarrow 0} \frac{1}{2} Q(\Delta_h \cdot, a)$  exists a.e. in  $t \in (-\infty, T]$ .

To discuss certain continuity properties of solutions of (1.1), it is important to know whether or not  $\int_{-\infty}^{\infty} \omega^2 \text{Re } \hat{a}(\omega) \|\hat{u}_t(\omega)\|^2 d\omega$  is continuous in  $t$  given that it exists. For this, we observe that  $\hat{u}_{t-h}(\omega)$  is generated from  $\hat{u}_t(\omega)$  by multiplying by  $e^{i\omega h}$  and applying the Hilbert transform. The question thus reduces to boundedness of the Hilbert transform in the norm induced by  $\int_{-\infty}^{\infty} (1 + \omega^2 \text{Re } \hat{a}(\omega)) \|\hat{u}_t(\omega)\|^2 d\omega$ . Using Theorem 6.2 of [6] (p. 255), we find

**Lemma 2.4:** Let  $T \in \mathbb{R}$  and  $u \in L^2((-\infty, T], H)$  be given. Assume that (2.7) holds,  $\int_{-\infty}^{\infty} \omega^2 \text{Re } \hat{a}(\omega) \|\hat{u}_T(\omega)\|^2 d\omega$  exists, and that the " $(A_2)$ -condition"

$$\sup_I \left( \frac{1}{|I|} \int_I (1 + \omega^2 \text{Re } \hat{a}(\omega)) d\omega \right) \cdot \left( \frac{1}{|I|} \int_I \frac{d\omega}{1 + \omega^2 \text{Re } \hat{a}(\omega)} \right) < \infty \quad (2.18)$$

holds, where the sup in (2.18) is taken over all intervals  $I \subset \mathbb{R}$ . Then,

$\int_{-\infty}^{\infty} \omega^2 \text{Re } \hat{a}(\omega) \|\hat{u}_t(\omega)\|^2 d\omega$  exists for all  $t \leq T$  and is continuous in  $t$ .

**Remark 2.2:** Condition (2.18) holds if  $\hat{a}'(\omega) \sim \omega^{-\alpha}$  as  $\omega \rightarrow \infty$ , with  $0 < \alpha < 1$ . This is essentially the case if  $a'(t) \sim t^{\alpha-1}$  as  $t \rightarrow 0$ . Such kernels are suggested by molecular

theories [5], [24], [28]. In this case, the norm generated by  $\int_{-\infty}^{\infty} (1 + \omega^2 \operatorname{Re} \hat{a}(\omega)) \|\hat{u}_t(\omega)\|^2 d\omega$  is equivalent to a fractional order Sobolev norm.

Our next lemma will be used to modify the global existence proof of Dafermos and Nohel [4].

**Lemma 2.5:** Assume that (2.7) holds. Then, for each  $\epsilon > 0$ , there exists a constant  $C(\epsilon)$  such that

$$\int_{-\infty}^t \int_{-\infty}^s a'(s-\tau) u(\tau) d\tau ds \leq \epsilon \int_{-\infty}^t \|u(\tau)\|^2 d\tau + C(\epsilon) Q(u, t, a) \quad (2.19)$$

$$\forall t \in (-\infty, T],$$

for every  $T \in \mathbb{R}$  and every  $u \in L^2((-\infty, T]; H)$ .

**Proof:** Taking Laplace transforms, (2.19) reduces to

$$|\hat{a}'(\omega)|^2 \leq \epsilon + C(\epsilon) \operatorname{Re} \hat{a}(\omega) \quad \forall \omega \in \mathbb{R}. \quad (2.20)$$

This last inequality is immediate since  $\operatorname{Re} \hat{a}(\omega) > 0$  and  $\lim_{|\omega| \rightarrow 0} |\hat{a}'(\omega)| = 0$  (by the Riemann-Lebesgue lemma). ■

**Remark 2.3:** If  $a'' \in L^1(0, \infty)$ , then (2.19) holds with  $\epsilon = 0$  and  $C(0) < \infty$ . This version of the lemma was used by Dafermos and Nohel [4].

We now discuss approximation of  $a$  by regular kernels. At this point, <sup>\*</sup> we assume

$$a, a' \in L^1(0, \infty), \quad (2.21)$$

$$a > 0, a' \leq 0, a'' > 0 \quad (\text{in the sense of measures}); \quad (2.22)$$

$$a'' \text{ is not a purely singular measure.}$$

<sup>\*</sup>

The problem of approximating an arbitrary strongly positive definite kernel by "regularized" strongly positive definite kernels does not appear to be easy. We could base our existence argument on an approximation method other than approximating the kernel, e.g. finite differences. If this is done, (2.22) is not needed, but the proofs become much more complicated. Moreover, (2.22) is a natural assumption from the viewpoint of applications to viscoelasticity.

As is well known, this implies that  $a$  is strongly positive definite. (Corollary 2.2 of [20].) For each  $\delta > 0$ , we define the approximating kernel  $a_\delta : [0, \infty) \rightarrow \mathbb{R}$  by

$$a_\delta(t) := \int_{-\delta}^{\delta} \rho_\delta(\tau) a(t+\delta-\tau) d\tau \quad \forall t \geq 0, \quad (2.23)$$

where  $\rho_\delta$  is a standard mollifier with support contained in  $[-\delta/2, \delta/2]$ .

It follows from (2.21), (2.22), (2.23) that for every  $\delta > 0$

$$a_\delta \in C^\infty[0, \infty), \quad a_\delta > 0, \quad a_\delta' < 0, \quad a_\delta'' > 0, \quad (2.24)$$

$$a_\delta, a_\delta', a_\delta'' \in L^1(0, \infty), \quad (2.25)$$

and

$$\|a_\delta\|_1 < \|a\|_1, \quad \|a_\delta'\|_1 < a(0), \quad (2.26)$$

where  $\|\cdot\|_1$  denotes the norm in  $L^1(0, \infty)$ . (Of course,  $\|a_\delta'\|_1$  does not necessarily remain bounded as  $\delta \rightarrow 0$ .) It also follows that  $a_\delta$  is strongly positive definite for  $\delta$  sufficiently small and that  $a_\delta \rightarrow a$  pointwise (and in  $L^1(0, \infty)$ ) as  $\delta \rightarrow 0$ . Moreover,  $\sup_{\omega \in \mathbb{R}} |\hat{a}_\delta(\omega)| < a(0)$  for all  $\delta > 0$ , and  $\operatorname{Re} \hat{a}_\delta \rightarrow \operatorname{Re} \hat{a}$  in  $L^1(\mathbb{R})$  as  $\delta \rightarrow 0$ . Therefore, a simple modification of the proof of Lemma 2.3 yields

**Lemma 2.6:** Assume that (2.21), (2.22) hold and let  $\varepsilon > 0$  be given. Then, there exist constants  $C(\varepsilon)$ ,  $\delta_0(\varepsilon) > 0$  such that for every  $\delta \in (0, \delta_0(\varepsilon)]$

$$\lim_{h \rightarrow 0} \frac{1}{2h} Q(\Delta_h u, t, a_\delta) > \left(\frac{1}{2} a(\delta) - \varepsilon\right) \|u(t)\|^2 \quad (2.27)$$

$$- C(\varepsilon) \int_{-\infty}^t \|u(s)\|^2 ds \quad \text{a.e. } t \in (-\infty, T],$$

for every  $T \in \mathbb{R}$  and every  $u \in L^2((-\infty, T]; H)$  such that  $\lim_{h \rightarrow 0} \frac{1}{2h} Q(\Delta_h u, t, a)$  exists a.e. in  $t \in (-\infty, T]$ .

In our subsequent use of this material, we shall always take  $H$  to be (the complexification of)  $L^2(0, 1)$ .

### 3. Linear Equations

In this section, we study the linear history value problem

$$u_{tt}(x,t) = \alpha(x,t)u_{xx}(x,t) + \int_{-\infty}^t a'(t-\tau)\beta(x,\tau)u_{xx}(x,\tau)d\tau + f(x,t), \quad (3.1)$$

$$x \in [0,1], t \in (-\infty, T],$$

$$u(0,t) = u(1,t) = 0, t \in (-\infty, T], \quad (3.2)$$

$$u(x,t) = v(x,t), x \in [0,1], t \in (-\infty, 0], \quad (3.3)$$

where  $T$  is a given positive number. We begin by stating an existence result for the case when the kernel does not have a singularity. There are many such existence theorems in the literature. (See, for example, [2], [10], and the references therein.) The particular one which we give here has been formulated with smoothness assumptions which are appropriate for our treatment of quasilinear equations in the next section.

We assume that the coefficients satisfy

$$\alpha, \alpha_x, \alpha_t, \alpha_{xx}, \alpha_{xt}, \alpha_{tt}, \beta, \beta_x, \beta_t, \beta_{xx}, \beta_{xt}, \beta_{tt} \in L^\infty((-\infty, T]; L^2(0,1)), \quad (3.4)$$

$$\alpha(x,t) > \underline{\alpha} > 0 \quad \forall x \in [0,1], t \in (-\infty, T]. \quad (3.5)$$

Of  $f$  and  $v$  we require

$$f, f_x, f_t \in L^\infty((-\infty, T]; L^2(0,1)) \cap L^2((-\infty, T]; L^2(0,1)), \quad (3.6)$$

$$f_{tt} \in L^2((-\infty, T]; L^2(0,1)),$$

$$v, v_x, v_t, v_{xx}, v_{xt}, v_{tt}, v_{xxx}, v_{xxt} \in L^\infty((-\infty, 0]; L^2(0,1)) \cap L^2((-\infty, 0]; L^2(0,1)). \quad (3.7)$$

In addition, we assume that  $v$  satisfies the equation and boundary conditions for  $t < 0$ , i.e.

$$v_{tt}(x,t) = \alpha(x,t)v_{xx}(x,t) + \int_{-\infty}^t a'(t-\tau)\beta(x,\tau)v_{xx}(x,\tau)d\tau + f(x,t), \quad (3.8)$$

$$x \in [0,1], t \in (-\infty, 0],$$

$$v(0,t) = v(1,t) = 0, t \in (-\infty, 0]. \quad (3.9)$$



Lemma 3.1: Assume that  $a', a'' \in L^1(0, \infty)$ ,  $\alpha$  and  $\beta$  satisfy (3.4), and that (3.5) holds for some constant  $\underline{\alpha} > 0$ . Let  $f$  and  $v$  satisfying (3.6) through (3.9) be given. Then, the history value problem (3.1), (3.2), (3.3) has a unique solution  $u$  with

$$u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in L^\infty((-\infty, T]; L^2(0, 1)) \quad (3.10)$$

If, in addition,

$$f_x \in C([0, T]; L^2(0, 1)) \quad (3.11)$$

then the solution has the additional regularity

$$u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in C([0, T]; L^2(0, 1)) \quad (3.12)$$

for positive time.

We have been unable to locate an existence theorem in the literature which has precisely the same smoothness conditions as Lemma 3.1. However, this type of result is standard and we omit the proof. For example, a minor modification of the proof of Theorem 2.1 of [4] can be used to establish Lemma 3.1.

We now prove an existence theorem which allows  $a'$  to have a singularity at 0. For this case, we must assume that the memory term satisfies the appropriate sign conditions, i.e. that (2.21), (2.22) hold and

$$\beta(x, t) > \underline{\beta} > 0 \quad \forall x \in [0, 1], t \in (-\infty, T] \quad (3.13)$$

Theorem 3.1: Assume that (2.21), (2.22), (3.4), (3.5), (3.13) hold, and let  $f$  and  $v$  satisfying (3.6) through (3.9) be given. Then, the history value problem (3.1), (3.2), (3.3) has a unique solution  $u$  which satisfies (3.10). If, in addition, (2.18) and (3.11) hold, then  $u$  has the additional regularity (3.12) for positive time.

Proof: Consider the family of approximating problems

$$u_{tt}^{(\delta)}(x, t) = \alpha(x, t)u_{xx}^{(\delta)}(x, t) + \int_{-\infty}^t a_\delta(t-\tau)\beta(x, \tau)u_{xx}^{(\delta)}(x, \tau)d\tau + f(x, t), \quad (3.14)$$

$$x \in [0, 1], t \in (-\infty, T]$$

$$u^{(\delta)}(0,t) = u^{(\delta)}(1,t) = 0, \quad t \in (-\infty, T], \quad (3.15)$$

$$u^{(\delta)}(x,t) = v(x,t), \quad x \in [0,1], \quad t \in (-\infty, 0], \quad (3.16)$$

for  $\delta > 0$ , where  $a_\delta$  is defined by (2.23). It follows from Lemma 3.1 that for each  $\delta > 0$ , (3.14), (3.15), (3.16) has a unique solution  $u^{(\delta)}$  with  $u^{(\delta)}, u_x^{(\delta)}, u_t^{(\delta)}, u_{xx}^{(\delta)}, u_{xt}^{(\delta)}, u_{tt}^{(\delta)}, u_{xxx}^{(\delta)}, u_{xxt}^{(\delta)}, u_{xtt}^{(\delta)}, u_{ttt}^{(\delta)} \in L^\infty((-\infty, T]; L^2(0,1))$ .

Our objective is to show that  $u^{(\delta)}$  obeys certain a priori bounds, uniformly in  $\delta$ , that imply the existence of a sequence  $\{u^{(\delta_n)}\}_{n=1}^\infty$  which converges to a solution as  $\delta_n \rightarrow 0$ . In order to simplify the notation, we suppress the superscript on  $u^{(\delta)}$ . For the purpose of deriving such bounds, we set

$$\begin{aligned} V := \operatorname{ess\,sup}_{s \in (-\infty, 0]} \int_0^1 \{v_x^2 + v_{xx}^2 + v_{xxx}^2 + v_{xxt}^2 + v_{xtt}^2 + v_{ttt}^2\}(x,s) dx \\ + \int_{-\infty}^0 \int_0^1 \{v_{xx}^2 + v_{xxx}^2 + v_{xxt}^2 + v_{xtt}^2 + v_{ttt}^2\}(x,s) dx ds, \end{aligned} \quad (3.17)$$

$$\begin{aligned} F := \operatorname{ess\,sup}_{s \in (-\infty, T]} \int_0^1 \{f_x^2 + f_t^2\}(x,s) dx \\ + \int_{-\infty}^T \int_0^1 \{f_x^2 + f_t^2 + f_{tt}^2\}(x,s) dx ds, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \Gamma_0 := \operatorname{ess\,sup}_{s \in (-\infty, 0]} \int_0^1 \{a^2 + a_x^2 + a_t^2 + a_{xx}^2 + a_{xt}^2 + a_{tt}^2 \\ + \beta^2 + \beta_x^2 + \beta_t^2 + \beta_{xx}^2 + \beta_{xt}^2 + \beta_{tt}^2\}(x,s) dx, \end{aligned} \quad (3.19)$$

$$\begin{aligned} \Gamma_1 := \operatorname{ess\,sup}_{s \in [0, T]} \int_0^1 \{a^2 + a_x^2 + a_t^2 + a_{xx}^2 + a_{xt}^2 + a_{tt}^2 + \beta^2 \\ + \beta_x^2 + \beta_t^2 + \beta_{xx}^2 + \beta_{xt}^2 + \beta_{tt}^2\}(x,s) dx, \end{aligned} \quad (3.20)$$

and

$$E[u](t) := \operatorname{ess\,sup}_{s \in [0, t]} \int_0^1 \{u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s) dx \quad (3.21)$$

and we observe that there exists a constant  $\lambda > 0$  such that

$$\frac{\alpha(x,t)}{\beta(x,t)} > \lambda \quad \forall x \in [0,1], \quad t \in (-\infty, T], \quad (3.22)$$

by virtue of (3.4), (3.5), (3.13).

An integration by parts in (3.14) yields

$$u_{tt} = \gamma^{(\delta)} u_{xx} + \int_{-\infty}^t a_{\delta}(t-\tau) [\beta u_{xx}]_t(x, \tau) d\tau + f, \quad (3.23)$$

where

$$\gamma^{(\delta)}(x, t) := \alpha(x, t) - a_{\delta}(0)\beta(x, t). \quad (3.24)$$

We apply the backward difference operator  $\Delta_h$  (in the time variable) to (3.23), thus obtaining

$$\Delta_h u_{tt} = \Delta_h [\gamma^{(\delta)} u_{xx}] + \int_{-\infty}^t a_{\delta}(t-\tau) \Delta_h [(\beta u_{xx})_t](x, \tau) d\tau + \Delta_h f. \quad (3.25)$$

Then, we multiply (3.25) by  $\Delta_h [(\beta u_{xx})_t]$  and integrate over  $[0, 1] \times (-\infty, t]$ ,  $t \in [0, T]$ .

After several integrations by parts, we divide by  $h^2$  and let  $h \rightarrow 0$ . The outcome of this tedious, but straightforward, computation is

$$\begin{aligned} & \frac{1}{2} \int_0^1 \{ \beta \gamma^{(\delta)} u_{xxt}^2 + \beta u_{xtt}^2 \}(x, t) dx + \lim_{h \rightarrow 0} \frac{1}{h^2} Q(\Delta_h [(\beta u_{xx})_t], t, a_{\delta}) \\ & + \int_0^1 \{ \beta \gamma_t^{(\delta)} u_{xx} u_{xxt} + \beta f_t u_{xxt} \}(x, t) dx \\ & = \int_{-\infty}^t \int_0^1 \{ \frac{3}{2} \beta \gamma_t^{(\delta)} u_{xxt}^2 - \frac{3}{2} \beta_t \gamma^{(\delta)} u_{xxt}^2 + \frac{1}{2} \beta_t u_{xtt}^2 \\ & - \beta_x u_{xtt} u_{ttt} + \beta \gamma_{tt}^{(\delta)} u_{xx} u_{xxt} + 2\beta_t u_{xxt} u_{ttt} + \beta_{tt} u_{xx} u_{ttt} \\ & - \beta_{tt} \gamma^{(\delta)} u_{xx} u_{xxt} - \beta_t \gamma_t^{(\delta)} u_{xx} u_{xxt} - \beta_{tt} \gamma_t^{(\delta)} u_{xx}^2 \\ & + \beta f_{tt} u_{xxt} - \beta_t f_t u_{xxt} - \beta_{tt} f_t u_{xx} \}(x, s) dx ds \\ & \text{a.e. } t \in (-\infty, T], \end{aligned} \quad (3.26)$$

where  $Q$  is defined by (2.1) with  $H = L^2(0, 1)$ .

It is not a priori evident that  $\lim_{h \rightarrow 0} \frac{1}{h^2} Q(\Delta_h [(\beta u_{xx})_t], t, a_{\delta})$  exists for a.e.  $t \in (-\infty, T]$ . However, all of the other limits involved in the derivation of (3.26) exist for a.e.  $t \in (-\infty, T]$ , and consequently so does the limit in question.

Using (3.5), (3.13), (3.24), Lemma 2.6 (with  $\varepsilon$  sufficiently small relative to  $\frac{\lambda}{\eta}$ ), and the algebraic inequality  $|AB| \leq \eta A^2 + \frac{1}{4\eta} B^2 \quad \forall \eta > 0$ , we find that the left hand side of (3.24) is bounded from below by

$$\begin{aligned}
& \int_0^1 \left( \frac{1}{4} \underline{\lambda} \underline{\beta}^2 u_{xxt}^2 + \frac{1}{2} \underline{\beta} u_{xtt}^2 \right) (x, t) dx \\
& - C \int_0^1 \left( (\alpha_t^2 + \beta_t^2) u_{xx}^2 + f_t^2 \right) (x, t) dx \\
& - C \int_{-\infty}^t \int_0^1 \left( \beta^2 u_{xxt}^2 + \beta_t^2 u_{xx}^2 \right) (x, s) dx ds \\
& \quad \forall t \in (-\infty, T], \delta \in (-\infty, \delta_0] ,
\end{aligned} \tag{3.27}$$

where  $C$  is a positive constant (which depends on  $\underline{\lambda}$  and  $\underline{\beta}$ .)

Differentiating (3.14) with respect to  $t$  and  $x$ , and splitting the convolution integrals, we obtain

$$\begin{aligned}
u_{ttt} &= \alpha u_{xxt} + \alpha_t u_{xx} + f_t + \int_{-\infty}^0 a'_\delta(t-\tau) [\beta v_{xxt} + \beta_t v_{xx}] (x, \tau) d\tau \\
&+ \int_0^t a'_\delta(t-\tau) [\beta u_{xxt} + \beta_t u_{xx}] (x, \tau) d\tau ,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\alpha u_{xxx} + \int_0^t a'_\delta(t-\tau) [\beta u_{xxx}] (x, \tau) d\tau &= u_{xtt} - \alpha_x u_{xx} - f_x \\
&- \int_{-\infty}^0 a'_\delta(t-\tau) [\beta v_{xxx} + \beta_x v_{xx}] (x, \tau) d\tau \\
&- \int_0^t a'_\delta(t-\tau) [\beta_x u_{xx}] (x, \tau) d\tau .
\end{aligned} \tag{3.29}$$

It follows easily from (3.28) that

$$\begin{aligned}
\int_0^1 u_{ttt}^2 (x, t) dx &\leq 7 \int_0^1 \left( \alpha^2 u_{xxt}^2 + \alpha_t^2 u_{xx}^2 + f_t^2 \right) (x, t) dx \\
&+ 7a(0)^2 \text{ess} - \sup_{s \in [0, T]} \int_0^1 \left( \beta^2 u_{xxt}^2 + \beta_t^2 u_{xx}^2 \right) (x, s) dx \\
&+ 7a(0)^2 \text{ess} - \sup_{s \in (-\infty, 0]} \int_0^1 \left( \beta^2 v_{xxt}^2 + \beta_t^2 v_{xx}^2 \right) (x, s) dx \\
&\quad \text{a.e. } t \in [0, T] .
\end{aligned} \tag{3.30}$$

Using Gronwall's inequality in (3.29), we obtain, after a simple computation,

$$\begin{aligned}
& \int_0^1 [\alpha u_{xxx}]^2 (x, t) dx \\
& \leq 6 \exp[2a(0)\underline{\lambda}^{-1}] \text{ess} - \sup_{s \in [0, T]} \int_0^1 \left( u_{xtt}^2 + \alpha_x^2 u_{xx}^2 + f_x^2 + a(0)^2 \beta_x^2 u_{xx}^2 \right) (x, s) ds \\
& + 6 \exp[2a(0)\underline{\lambda}^{-1}] \text{ess} - \sup_{s \in (-\infty, 0]} \int_0^1 \left( \beta^2 v_{xxx}^2 + \beta_x^2 v_{xx}^2 \right) (x, s) dx \\
& \quad \text{a.e. } t \in [0, T] .
\end{aligned} \tag{3.31}$$

Combining (3.26), (3.30), and (3.31), and recalling the lower bound (3.27), we conclude that there exists a positive constant  $K$  such that

$$E[u](t) \leq K[F + (1 + \Gamma_0 + \Gamma_1 T)V] + K \cdot (1 + \Gamma_1) \cdot (1 + T^2) \int_0^t E[u](s) ds \quad (3.32)$$

$$\forall t \in [0, T], \delta \in (0, \delta_0] .$$

(The constant  $K$  depends on  $\underline{\alpha}, \underline{\beta}, \underline{\lambda}$ , and  $a$ , but is independent of  $F, V, \Gamma_0, \Gamma_1, T$ , and  $\delta$ .) Gronwall's inequality and (3.32) yield

$$E[u](T) \leq K[F + (1 + \Gamma_0 + \Gamma_1 T)V] \exp[K \cdot (1 + \Gamma_1) \cdot (T + T^3)] \quad (3.33)$$

for all  $\delta \in (0, \delta_0]$ .

To assist the reader in following the derivation of (3.32), we show the detailed estimation of a few typical terms. By the Sobolev embedding theorem,  $\beta_x^2(x, t) \leq \Gamma_0$  for all  $x \in [0, 1], t \in (-\infty, 0]$ , and  $\beta_x^2(x, t) \leq \Gamma_1$  for all  $x \in [0, 1], t \in [0, T]$ . Therefore,

$$\begin{aligned} \left| \int_{-\infty}^t \int_0^1 \beta_x u_{xtt} u_{ttt}(x, s) dx ds \right| &\leq \frac{1}{2} \int_{-\infty}^t \int_0^1 \{\beta_x^2 u_{xtt}^2 + u_{ttt}^2\}(x, s) dx ds \\ &= \frac{1}{2} \int_{-\infty}^0 \int_0^1 \{\beta_x^2 v_{xtt}^2 + v_{ttt}^2\}(x, s) dx ds \\ &\quad + \frac{1}{2} \int_0^t \int_0^1 \{\beta_x^2 u_{xtt}^2 + u_{ttt}^2\}(x, s) dx ds \\ &\leq \frac{1}{2} (\Gamma_0 + 1)V + \frac{1}{2} (\Gamma_1 + 1)E[u](t) \quad \forall t \in [0, T] . \end{aligned} \quad (3.34)$$

Next, we observe that

$$\max_{\xi \in [0, 1]} v_{xx}^2(\xi, s) \leq \int_0^1 \{v_{xx}^2 + v_{xxx}^2\}(x, s) dx \quad \forall s \in (-\infty, 0] , \quad (3.35)$$

and consequently

$$\int_{-\infty}^0 \max_{\xi \in [0, 1]} v_{xx}^2(\xi, s) ds \leq V . \quad (3.36)$$

In addition, we note that

$$u_{xx}(x, t) = v_{xx}(x, 0) + \int_0^t u_{xxt}(x, s) ds \quad \forall x \in [0, 1], t \in [0, T] , \quad (3.37)$$

from which we easily deduce the estimates

$$\begin{aligned} \int_0^1 u_{xx}^2(x, t) dx &\leq 2 \int_0^1 v_{xx}^2(x, 0) dx + 2t \int_0^t \int_0^1 u_{xxt}^2(x, s) dx ds \\ &\leq 2V + 2T^2 E[u](t) \quad \forall t \in [0, T] , \end{aligned} \quad (3.38)$$

and

$$\max_{x \in [0, 1]} u_{xx}^2(x, t) \leq 2V + (1 + 2T^2)E[u](t) \quad \forall t \in [0, T] . \quad (3.39)$$

Using (3.36) and (3.39), we find

$$\begin{aligned}
\left| \int_{-\infty}^t \int_0^1 \beta_{tt} u_{xx} u_{ttt}(x,s) dx ds \right| &< \frac{1}{2} \int_{-\infty}^t \int_0^1 (\beta_{tt}^2 u_{xx}^2 + u_{ttt}^2)(x,s) dx ds \\
&< \frac{1}{2} \int_{-\infty}^t \max_{\xi \in [0,1]} u_{xx}^2(\xi,s) \int_0^1 \beta_{tt}^2(x,s) dx ds + \frac{1}{2} \int_{-\infty}^t \int_0^1 u_{ttt}^2(x,s) dx ds \\
&< \frac{1}{2} \int_{-\infty}^0 \max_{\xi \in [0,1]} v_{xx}^2(\xi,s) \int_0^1 \beta_{tt}^2(x,s) dx ds + \frac{1}{2} \int_{-\infty}^0 \int_0^1 v_{ttt}^2(x,s) dx ds \\
&\quad + \frac{1}{2} \int_0^t \max_{\xi \in [0,1]} u_{xx}^2(\xi,s) \int_0^1 \beta_{tt}^2(x,s) dx ds + \frac{1}{2} \int_0^t \int_0^1 u_{ttt}^2(x,s) dx ds \\
&< \frac{1}{2} (\Gamma_0 + 1)V + \Gamma_1 VT + \frac{1}{2} \Gamma_1 (1 + 2T^2) \int_0^t E(u)(s) ds + \frac{1}{2} \int_0^t E(u)(s) ds
\end{aligned} \tag{3.40}$$

$\forall t \in [0, T]$

The other terms can all be handled in a similar manner.

We conclude from (3.33) that  $u_{xxx}^{(\delta)}$ ,  $u_{xxt}^{(\delta)}$ ,  $u_{xtt}^{(\delta)}$ , and  $u_{ttt}^{(\delta)}$  are bounded in  $L^\infty([0, T]; L^2(0, 1))$  independently of  $\delta \in (0, \delta_0]$ . It follows from (3.38) (and similar inequalities for the other derivatives) that  $u_{xx}^{(\delta)}$ ,  $u_{xt}^{(\delta)}$ ,  $u_{tt}^{(\delta)}$ ,  $u_x^{(\delta)}$ ,  $u_t^{(\delta)}$  and  $u^{(\delta)}$  are also bounded in  $L^\infty([0, T]; L^2(0, 1))$  independently of  $\delta \in (0, \delta_0]$ . Therefore, there exists a function  $u : [0, 1] \times (-\infty, T] \rightarrow \mathbb{R}$ , with  $u = v$  on  $[0, 1] \times (-\infty, 0]$ , and a sequence  $\{\delta_n\}_{n=1}^\infty$ , with  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that

$$\begin{aligned}
&u^{(\delta_n)}, u_x^{(\delta_n)}, u_t^{(\delta_n)}, u_{xx}^{(\delta_n)}, u_{xt}^{(\delta_n)}, u_{tt}^{(\delta_n)}, u_{xxx}^{(\delta_n)}, u_{xxt}^{(\delta_n)}, \\
&u_{xtt}^{(\delta_n)}, u_{ttt}^{(\delta_n)} \rightarrow u, u_x, u_t, \text{ etc. weakly star in } L^\infty([0, T]; L^2(0, 1))
\end{aligned} \tag{3.41}$$

as  $n \rightarrow \infty$ . Standard embedding theorems and (3.41) imply

$$\begin{aligned}
&u^{(\delta_n)}, u_x^{(\delta_n)}, u_t^{(\delta_n)}, u_{xx}^{(\delta_n)}, u_{xt}^{(\delta_n)}, u_{tt}^{(\delta_n)} \rightarrow u, u_x, u_t, u_{xx}, u_{xt}, u_{tt} \\
&\text{uniformly on } [0, 1] \times [0, T]
\end{aligned} \tag{3.42}$$

as  $n \rightarrow \infty$ . It thus follows easily that  $u$  satisfies (3.1), (3.2), (3.3).

Suppose that (2.18) and (3.11) hold. To show that the third order derivatives of  $u$  belong to  $C([0, T]; L^2(0, 1))$ , we argue along the lines of Strauss [27]. We first note that Theorem 2.1 of [27] implies that  $u_{xxx}$ ,  $u_{xxt}$ ,  $u_{xtt}$  and  $u_{ttt}$  are weakly continuous from  $(-\infty, T]$  to  $L^2(0, 1)$ . Then, the basic idea is to show that a certain energy which

acts like a "variable norm" of third derivatives is continuous. This, in conjunction with the aforementioned weak continuity, will imply the desired strong continuity.

We apply the procedure used to derive (3.26) to (3.1), (3.2), (3.3). We thus conclude that for a.e.  $t \in (-\infty, T]$ ,  $u$  satisfies (3.26) with  $a_0$  replaced by  $a$ . Using Lemmas 2.1 and 2.4, and the fact that the right hand side of (3.26) is continuous in  $t$ , we find that

$$\begin{aligned} H[u](t) := & \frac{1}{2} \int_0^1 \{ \beta \gamma u_{xxt}^2 + \beta u_{xxt}^2 \}(x, t) dx + \lim_{h \rightarrow 0} \frac{1}{2} Q(A_h[(\beta u_{xx})_t], t, a) \\ & + \int_0^1 \{ \beta \gamma_t u_{xx} u_{xxt} + \beta f_t u_{xxt} \}(x, t) dx \end{aligned} \quad (3.43)$$

is continuous on  $t$ . (Observe that  $H[u]$  is coercive in  $u_{xxt}$  and  $u_{ttt}$ , and that  $f_t \in C((-\infty, T]; L^2(0, 1))$  by (3.6).) A minor modification of the proof of Theorem 4.2 of [27] yields

$$u_{xxt}, u_{xtt} \in C((-\infty, T]; L^2(0, 1)) . \quad (3.44)$$

Differentiating (3.1) with respect to  $x$  and  $t$ , and using (3.44), we conclude that

$$u_{xxx} \in C([0, T]; L^2(0, 1)) , \quad (3.45)$$

and

$$u_{ttt} \in C((-\infty, T]; L^2(0, 1)) . \quad (3.46)$$

It is interesting to note that (3.44) and (3.46) hold even without the assumption (3.11). In particular if  $v$  satisfies (3.7), (3.8), and (3.9), it automatically satisfies  $v_{xxt}$ ,  $v_{xtt}$ ,  $v_{ttt} \in C((-\infty, 0]; L^2(0, 1))$ . Moreover, if  $f_x$  belongs to  $C((-\infty, 0]; L^2(0, 1))$ , then so does  $v_{xxx}$ . Finally, we note that the a priori bound (3.33) also holds for the "exact solution"  $u$ . ■

#### 4. Local Existence

We now apply the results of the preceding section to establish a local existence theorem for the quasilinear history value problem (1.1), (1.2), (1.3).

Theorem 4.1: Assume that  $\phi, \psi \in C^3(\mathbb{R})$ , (2.21) and (2.22) hold, and that

$$\phi'(\xi) > \underline{\phi} > 0, \psi'(\xi) > \underline{\psi} > 0 \quad \forall \xi \in \mathbb{R}. \quad (4.1)$$

Assume further that  $f$  satisfies (3.6) for every  $T > 0$ ,  $v$  satisfies (3.7), and that equations (1.1), (1.2) hold (with  $u = v$ ) for  $t < 0$ . Then, the history value problem (1.1), (1.2), (1.3) has a unique solution  $u$  defined on a maximal time interval  $(-\infty, T_0)$ ,  $T_0 > 0$ , which satisfies (3.10) for every  $T < T_0$ . If, in addition, (2.18) holds and

$$f_x \in C([0, \infty); L^2(0, 1)) \quad , \quad (4.2)$$

then (3.12) holds for every  $T \in (0, T_0)$ . Moreover, if

$$\begin{aligned} \text{ess - sup}_{t \in (-\infty, T_0)} \int_0^1 \{ & u^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 \\ & + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2 \} (x, t) dx < \infty \quad , \end{aligned} \quad (4.3)$$

then  $T_0 = \infty$ .

Proof: For each  $M, T > 0$ , let  $Z(M, T)$  denote the set of all functions

$w : [0, 1] \times (-\infty, T] \rightarrow \mathbb{R}$  such that

$$w, w_x, w_t, w_{xx}, w_{xt}, w_{tt}, w_{xxx}, w_{xxt}, w_{xtt}, w_{ttt} \in L^\infty((-\infty, T]; L^2(0, 1)) \quad , \quad (4.4)$$

$$w(0, t) = w(1, t) = 0 \quad \forall t \in (-\infty, T] \quad , \quad (4.5)$$

$$w(x, t) = v(x, t) \quad \forall x \in [0, 1], t \in (-\infty, 0] \quad , \quad (4.6)$$

and

$$\text{ess - sup}_{t \in [0, T]} \int_0^1 \{ w_{xxx}^2 + w_{xxt}^2 + w_{xtt}^2 + w_{ttt}^2 \} (x, t) dx < M \quad . \quad (4.7)$$

We note that  $Z(M, T)$  is nonempty for  $M$  sufficiently large. Henceforth, we tacitly make this assumption.



It follows from (4.1) that  $\inf_{\xi \in \mathbb{R}} [\phi'(\xi)/\psi'(\xi)] > 0$ . We temporarily make the stronger assumption

$$\underline{\nu} := \inf_{\xi \in \mathbb{R}} \frac{\phi'(\xi)}{\psi'(\xi)} > 0, \quad (4.8)$$

which will be removed later. Identifying  $\alpha$  with  $\phi'(w_x)$  and  $\beta$  with  $\psi'(w_x)$ , it follows immediately from Theorem 3.1 that for  $w \in Z(M, T)$ , the history value problem

$$u_{tt}(x, t) = \phi'(w_x) u_{xx}(x, t) + \int_{-\infty}^t a'(t-\tau) \psi'(w_x) u_{xx}(x, \tau) d\tau + f(x, t), \quad (4.9)$$

$$x \in [0, 1], \quad t \in (-\infty, T],$$

(1.2), (1.3) has a unique solution  $u$  which satisfies (3.10). Moreover, the corresponding  $\underline{\alpha}$ ,  $\underline{\beta}$ , and  $\underline{\lambda}$  can be chosen independently of  $M$  and  $T$ .

Let  $S$  denote the mapping which carries  $w$  into the solution of (4.9), (1.2), (1.3). Our goal is to show that, for appropriately chosen  $M$  and  $T$ ,  $S$  has a unique fixed point in  $Z(M, T)$  which is obviously a solution of (1.1), (1.2), (1.3). For this purpose, we employ the contraction mapping principle and the complete\* metric  $\rho$  given by

$$\rho(w, \bar{w}) := \max_{t \in [0, T]} \left( \int_0^1 \{ (w_{xx} - \bar{w}_{xx})^2 + (w_{xt} - \bar{w}_{xt})^2 + (w_{tt} - \bar{w}_{tt})^2 \} (x, t) dx \right). \quad (4.10)$$

Observe that for  $w \in Z(M, T)$ , we have

$$w_{xx}(x, t) = v_{xx}(x, 0) + \int_0^t w_{xxt}(x, s) ds \quad \forall x \in [0, 1], \quad t \in [0, T]. \quad (4.11)$$

Therefore,

$$\begin{aligned} \int_0^1 w_{xx}^2(x, t) dx &\leq 2 \int_0^1 v_{xx}^2(x, 0) dx + 2t \int_0^t \int_0^1 w_{xxt}^2(x, s) ds \\ &\leq 2V + 2Mt^2 \quad \forall t \in [0, T], \end{aligned} \quad (4.12)$$

where  $V$  is defined by (3.17), and so clearly

$$\sup_{t \in [0, T]} \int_0^1 w_{xx}^2(x, t) dx \leq 2V + 2MT^2 \quad \forall w \in Z(M, T). \quad (4.13)$$

Similarly, the following inequalities hold for all  $w \in Z(M, T)$ :

$$\sup_{t \in [0, T]} \int_0^1 w_{xt}^2(x, t) dx \leq 2V + 2MT^2, \quad (4.14)$$

---

\* Completeness of  $\rho$  follows from Alouglu's theorem and sequential weak star lower semi-continuity of the norm in  $L^\infty([0, T]; L^2(0, 1))$ .

$$\sup_{\substack{x \in [0,1] \\ t \in [0,T]}} w_{xx}^2(x,t) < 2V + (1 + 2T^2)M, \quad (4.15)$$

$$\sup_{\substack{x \in [0,1] \\ t \in [0,T]}} w_{xt}^2(x,t) < 2V + (1 + 2T^2)M, \quad (4.16)$$

$$\sup_{\substack{x \in [0,1] \\ t \in [0,T]}} w_x^2(x,t) < 2V(1 + 2T^2) + (2T^2 + 4T^4)M. \quad (4.17)$$

The a priori estimate (3.33) and the above inequalities show that  $S$  maps  $Z(M,T)$  into itself provided that  $T$  is sufficiently small relative to  $M$ . From now on, we assume that  $T$  is small enough so that  $S$  maps  $Z(M,T)$  into  $Z(M,T)$ .

To show that  $S$  is a contraction, let  $M, T > 0$  and  $w, \bar{w} \in Z(M,T)$  be given, and set  $u := Sw$ ,  $\bar{u} := S\bar{w}$ ,  $w := w - \bar{w}$ ,  $U := u - \bar{u}$ . A simple computation shows that  $U$  satisfies

$$\begin{aligned} U_{tt} = & \phi'(w_x)U_{xx} + \int_0^t a'(t-\tau)\phi'(w_{xx})U_{xx}(x,\tau)d\tau + [\phi'(w_x) - \phi'(\bar{w}_x)]\bar{u}_{xx} \\ & + \int_0^t a'(t-\tau)[\psi'(w_x) - \psi'(\bar{w}_x)]\bar{u}_{xx}(x,\tau)d\tau, \quad \forall x \in [0,1], t \in [0,T], \end{aligned} \quad (4.18)$$

$$U(0,t) = U(1,t) = 0, \quad \forall t \in [0,T] \quad (4.19)$$

$$U(x,t) = 0 \quad \forall x \in [0,1], t \in (-\infty, 0]. \quad (4.20)$$

Integrating the first convolution term in (4.18) by parts, we obtain

$$\begin{aligned} U_{tt} = & \chi'(w_x)U_{xx} + \int_0^t a(t-\tau)[\psi'(w_x)U_{xx}]_t(x,\tau)d\tau + [\phi'(w_x) - \phi'(\bar{w}_x)]\bar{u}_{xx} \\ & + \int_0^t a'(t-\tau)[\psi'(w_x) - \psi'(\bar{w}_x)]\bar{u}_{xx}(x,\tau)d\tau, \end{aligned} \quad (4.21)$$

where

$$\chi(\xi) := \phi(\xi) - a(0)\psi(\xi) \quad \forall \xi \in \mathbb{R}. \quad (4.22)$$

We multiply (4.21) by  $[\phi'(w_x)U_{xx}]_t$  and integrate over  $[0,1] \times [0,t]$ ,  $t \in [0,T]$ , performing various integrations by parts and exploiting (4.19), (4.20). This yields

$$\begin{aligned}
& \frac{1}{2} \int_0^1 \{ \psi'(w_x) \chi'(w_x) U_{xx}^2 + \psi'(w_x) U_{xt}^2 \} (x,t) dx + Q(\psi'(w_x) U_{xx})_t, t, a) \\
& = - \int_0^1 [\phi'(w_x) - \phi'(\bar{w}_x)] \psi'(w_x) \bar{u}_{xx} U_{xx} (x,t) dx \\
& - \int_0^1 \psi'(w_x) U_{xx} (x,t) \int_0^t a'(t-\tau) [\psi'(w_x) - \psi'(\bar{w}_x)] \bar{u}_{xx} (x,\tau) d\tau dx \\
& + \int_0^t \int_0^1 \left\{ \frac{1}{2} \psi''(w_x) w_{xt} U_{xt}^2 - \psi''(w_x) w_{xx} U_{xt} U_{tt} + \psi''(w_x) w_{xt} U_{xx} U_{tt} \right. \\
& \quad + \frac{1}{2} [\chi''(w_x) \psi'(w_x) - \chi'(w_x) \psi''(w_x)] w_{xt} U_{xx}^2 \\
& \quad + [\phi'(w_x) - \phi'(\bar{w}_x)] \psi'(w_x) \bar{u}_{xxt} U_{xx} \\
& \quad + [\phi''(w_x) - \phi''(\bar{w}_x)] \psi'(w_x) \bar{u}_{xx} U_{xx} w_{xt} \} (x,s) dx ds \\
& + \int_0^t \int_0^1 \psi'(w_x) U_{xx} (x,s) \int_0^s a'(s-\tau) [\psi'(w_x) - \psi'(\bar{w}_x)] \bar{u}_{xxt} \\
& \quad + [\psi''(w_x) - \psi''(\bar{w}_x)] \bar{u}_{xx} w_{xt} \} (x,\tau) d\tau dx ds \quad \forall t \in [0,T] .
\end{aligned} \tag{4.23}$$

Using (4.1) and Lemma 2.3 with  $\varepsilon$  sufficiently small, we see that the left hand side of (4.23) is bounded from below by

$$\begin{aligned}
& \int_0^1 \left\{ \frac{1}{4} U_{xx}^2 + \frac{1}{2} U_{xt}^2 \right\} (x,t) \\
& - C \int_0^t \int_0^1 [\psi'(w_x) U_{xx}^2] (x,s) dx ds \quad \forall t \in [0,T] ,
\end{aligned} \tag{4.24}$$

where  $C$  is a constant that can be chosen independently of  $M$  and  $T$ .

It follows from (4.18) that

$$\begin{aligned}
& \int_0^1 U_{tt}^2 (x,t) dt \leq 4 \int_0^1 \{ \phi'(w_x)^2 U_{xx}^2 + [\phi'(w_x) - \phi'(\bar{w}_x)]^2 \bar{u}_{xx}^2 \} (x,t) dx \\
& + 4a(0)^2 \max_{s \in [0,t]} \int_0^1 \{ \psi'(w_x)^2 U_{xx}^2 + [\psi'(w_x) - \psi'(\bar{w}_x)]^2 \bar{u}_{xx}^2 \} (x,s) dx \\
& \quad \forall t \in [0,T] .
\end{aligned} \tag{4.25}$$

We combine (4.23) and (4.25) and proceed as in the derivation of (3.33). Exploiting the fact that  $w \equiv 0$  on  $[0,1] \times (-\infty, 0]$ , we obtain (after a rather long computation) an estimate of the form

$$\rho(Sw, S\bar{w}) \leq P(M, T) \exp(T \cdot Q(M, T)) \rho(w, \bar{w}) \quad \forall w, \bar{w} \in Z(M, T) \tag{4.26}$$

for every  $M, T > 0$ , where  $P, Q : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  are continuous functions with  $P(M, 0) = 0 \quad \forall M > 0$ .

The derivation of (4.26) from (4.23) and (4.25) is in much the same spirit as the derivation of (3.33). We show the detailed estimation of the first term on the right hand side of (4.25). For each  $\eta > 0$ , we have

$$\begin{aligned} \left| \int_0^1 [\phi'(w_x) - \phi'(\bar{w}_x)] \psi'(w_x) \bar{u}_{xx} u_{xx}(x,t) dx \right| &< \eta \int_0^1 u_{xx}^2(x,t) dx \\ &+ (4\eta)^{-1} \int_0^1 [\phi'(w_x) - \phi'(\bar{w}_x)]^2 \psi'(w_x)^2 \bar{u}_{xx}^2(x,t) dx \quad \forall t \in [0, T] . \end{aligned} \quad (4.27)$$

If we choose  $\eta$  sufficiently small, the first integral on the right hand side of (4.27) can be absorbed by the first integral in (4.24). To estimate the last integral in (4.27), we first observe that by (4.17) and the mean value theorem

$$[\phi'(w_x) - \phi'(\bar{w}_x)]^2(x,t) \leq \Phi(M,T) w_x^2(x,t) \quad \forall x \in [0,1], t \in [0,T] , \quad (4.28)$$

where  $\Phi(M,T) := \max \phi''(\xi)^2$  and the max is taken over all  $\xi$  with  $\xi^2 \leq 2V(1+2T^2) + (2T^2+4T^4)M$ . Using the fact that  $w \equiv 0$  on  $[0,1] \times (-\infty, 0]$ , the type of argument used to derive (4.17) yields

$$w_x^2(x,t) \leq 4M(T^2+T^4) \quad \forall x \in [0,1], t \in [0,T] . \quad (4.29)$$

Next, we set  $V(M,T) := \max \psi'(\xi)^2$  where the max is taken over all  $\xi$  with  $\xi^2 \leq 2V(1+2T^2) + (2T^2+4T^4)M$ . Then, using (4.13), (4.28), and the fact that  $\bar{u} \in Z(M,T)$ , we find

$$\begin{aligned} \int_0^1 [\phi'(w_x) - \phi'(\bar{w}_x)]^2 \psi'(w_x)^2 \bar{u}_{xx}^2(x,t) dx \\ \leq 8M(T^2+T^4) \Phi(M,T) V(M,T) (V + MT^2) \quad \forall t \in [0,T] . \end{aligned} \quad (4.30)$$

The remaining steps in the derivation of (4.26) can be carried out in a similar fashion.

The contraction mapping principle and (4.26) imply that  $S$  has a unique fixed point  $u \in Z(M,T)$  for a sufficiently small choice of  $T > 0$ . It is obvious that  $u$  satisfies (1.1), (1.2), (1.3) on  $[0,1] \times (-\infty, T]$ . The uniqueness statement in Theorem 4.1 is immediate. If (2.18) and (4.2) hold, the additional regularity (3.12) follows from Theorem 3.1 and the fact that  $u$  satisfies (4.8), (1.2), (1.3) with  $w = u$ . The continuation of  $u$  to a maximal time interval  $(-\infty, T_0)$  with the property that (4.3) implies  $T_0 = \infty$  follows from essentially the same argument as in [4].

It is easy to remove the extraneous assumption (4.8). To do so, we construct a function  $\tilde{\psi} \in C^3(\mathbb{R})$  which satisfies

$$\tilde{\psi}(\xi) = \psi(\xi) \quad \forall \xi \in [-2\sqrt{v}, 2\sqrt{v}] \quad , \quad (4.31)$$

$$\inf_{\xi \in \mathbb{R}} \tilde{\psi}'(\xi) > 0, \quad \sup_{\xi \in \mathbb{R}} \tilde{\psi}'(\xi) < \infty \quad , \quad (4.32)$$

and we consider equation (1.1) with  $\psi$  replaced by  $\tilde{\psi}$ . The preceding argument shows that the modified history value problem has a unique solution  $u$  on  $(-\infty, T]$  for some  $T > 0$ . The Sobolev embedding theorem implies that

$$\sup_{\substack{x \in [0,1] \\ t \in (-\infty, 0]}} v_x^2(x, t) < v \quad . \quad (4.33)$$

By virtue of (4.31), (4.33), and the continuity properties of  $u_x$ ,  $u$  is a solution of the original problem on some smaller interval  $(-\infty, \bar{T}]$  with  $\bar{T} > 0$ . The additional properties of  $u$  as a solution of the original problem all follow easily. ■

## 5. Global Existence

The following result is a precise analogue of Theorem 4.1 in Dafermos and Nohel [4].

Recall that  $\chi(\xi) := \phi(\xi) - a(0)\psi(\xi) \quad \forall \xi \in \mathbb{R}$ .

**Theorem 5.1:** Let the following assumptions hold:

- (i)  $a, a' \in L^1(0, \infty)$ ,  $a > 0$ ,  $a' < 0$ ,  $a'' > 0$  (in the sense of measures);  $a''$  is not a purely singular measure,
- (ii)  $\phi, \psi \in C^3$ ,  $\phi(0) = \psi(0) = 0$ ,  $\phi'(0) > 0$ ,  $\psi'(0) > 0$ ,  $\chi'(0) > 0$ ,
- (iii)  $f, f_t, f_x \in L^\infty((-\infty, \infty); L^2(0, 1)) \cap L^2((-\infty, \infty); L^2(0, 1))$   
 $f_{tt} \in L^2((-\infty, \infty); L^2(0, 1))$ , and the norms of  $f, f_t, f_x, f_{tt}$  in the indicated spaces are sufficiently small,
- (iv) The given history for  $t < 0$  satisfies the equation and boundary conditions and all derivatives through third order lie in  $L^\infty((-\infty, 0]; L^2(0, 1)) \cap L^2((-\infty, 0]; L^2(0, 1))$ .

Then, (1.1), (1.2), (1.3) has a unique solution  $u$  existing for all  $t \in (-\infty, \infty)$  such that  $u$  and all derivatives through third order lie in  $L^\infty((-\infty, \infty); L^2(0, 1)) \cap L^2((-\infty, \infty); L^2(0, 1))$ . Moreover,  $u$  and derivatives through second order converge to zero uniformly as  $t \rightarrow \infty$ . If, in addition, the  $(A_2)$ -condition (2.18) holds and  $f_x \in C([0, \infty), L^2(0, 1))$ , then third derivatives of  $u$  belong to  $C([0, \infty), L^2(0, 1))$ .

The proof is essentially a line-by-line copy of the argument of Dafermos and Nohel.

We need only note that in deriving their estimate (3.26) they use Lemma 2.5 with

$\varepsilon = 0$ , while we have to use Lemma 2.5 with  $\varepsilon \neq 0$  but small. Apart from this simple change, their proof goes through unaltered.

### Remarks:

- 5.1: In assumption (iv), we did not require smallness of the norms. However, this smallness is implied by assumption (iii) and the fact that  $v$  satisfies the equation and boundary conditions for  $t < 0$ .

5.2: Theorem 5.1 applies without essential changes if Dirichlet conditions are replaced by Neumann or mixed conditions. In the case of Neumann conditions, the boundedness and decay statements apply to  $u$  minus its spatial mean value  $\langle u \rangle$  which evolves according to the trivial equation

$$\frac{d^2}{dt^2} \langle u \rangle(t) = \langle f \rangle(t) .$$

5.3: The question of global existence for the all-space problem is more difficult. Hrusa and Nohel [13] gave a proof for regular kernels. This proof, however, makes essential use of the assumption  $a'' \in L^1(0, \infty)$  and does not appear generalizable to singular kernels.

5.4: It would be interesting, if a global existence result could be established assuming only  $\chi' > 0$  in a neighborhood of 0 rather than  $\chi'(0) > 0$ . Even for regular kernels, this has been accomplished only for the case  $\chi' \equiv 0$  which arises in modelling shear flows of viscoelastic fluids and in models for heat flow in materials with memory. (See [3], [17], and [26].) The global estimates of Dafermos and Nohel [4], which, as remarked, can be carried out without assuming  $a'' \in L^1$ , can also be adapted to  $\chi' \equiv 0$ , without assuming  $a'' \in L^1(0, \infty)$ . However, the hypotheses on  $f$  in this case must be slightly different than those above.

5.5: It is conceivable that, for an appropriate class of singular kernels, global smooth solutions exist even for large data. However, we have not been able to verify this.

# References

- [1] B. Bernstein and R. R. Huilgol, On ultrasonic dynamic moduli, Trans. Soc. Rheology 18 (1974), 583-590.
- [2] C. M. Dafermos, An abstract Volterra equation with applications to linear viscoelasticity, J. Diff. Eq. 7 (1970), 554-569.
- [3] C. M. Dafermos and J. A. Nohel, Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, Comm. PDE 4 (1979), 219-278.
- [4] C. M. Dafermos and J. A. Nohel, A nonlinear hyperbolic Volterra equation in viscoelasticity, Amer. J. Math., Supplement (1981), 87-116.
- [5] M. Doi and S. F. Edwards, Dynamics of concentrated polymer systems, J. Chem. Soc. Faraday 74 (1978), 1789-1832 and 75 (1979), 38-54.
- [6] J. B. Garnett, Bounded Analytic Functions, Academic Press, New York 1981.
- [7] G. Gripenberg, Nonexistence of smooth solutions for shearing flows in a nonlinear viscoelastic fluid, SIAM J. Math. Anal. 13 (1982), 954-961.
- [8] K. B. Hannsgen and R. L. Wheeler, Behavior of the solutions of a Volterra equation as a parameter tends to infinity, J. Integral Equations (to appear).
- [9] H. Hattori, Breakdown of smooth solutions in dissipative nonlinear hyperbolic equations, Quart. Appl. Math. 40 (1982/83), 113-127.
- [10] M. L. Heard, Existence theorems for hyperbolic Volterra integrodifferential equations, (preprint).
- [11] W. J. Hrusa, A nonlinear functional differential equation in Banach space with applications to materials with fading memory, Arch. Rational Mech. Anal. 84 (1983), 99-137.
- [12] W. J. Hrusa and J. A. Nohel, Global existence and asymptotics in one-dimensional nonlinear viscoelasticity, Proc. 5th Symp. on Trends in Appl. of Pure Math. to Mech., Springer Lecture Notes in Physics.
- [13] W. J. Hrusa and J. A. Nohel, The Cauchy problem in one-dimensional nonlinear viscoelasticity, J. Differential Equations (submitted).



- [14] J. U. Kim, Global smooth solutions for the equations of motion of a nonlinear fluid with fading memory, Arch. Rational Mech. Anal. 79 (1982), 97-130.
- [15] H. M. Laun, Description of the non-linear shear behavior of a low density polyethylene melt by means of an experimentally determined strain dependent memory function, Rheol. Acta 17 (1978), 1-15.
- [16] S.-O. Londen, An existence result on a Volterra equation in a Banach space, Trans. Amer. Math. Soc. 235 (1978), 285-304.
- [17] R. C. MacCamy, A model for one-dimensional nonlinear viscoelasticity, Q. Appl. Math. 35 (1977), 21-33.
- [18] R. Malek-Madani and J. A. Nohel, Formation of singularities for a conservation law with memory, SIAM J. Math. Anal.
- [19] P. A. Markowich and M. Renardy, Lax-Wendroff methods for hyperbolic history value problems, SIAM J. Num. Anal. 21 (1984), 24 - 51.
- [20] J. A. Nohel and D. F. Shea, Frequency domain methods for Volterra equations, Adv. Math. 22 (1976), 278-304.
- [21] M. Renardy, Singularly perturbed hyperbolic evolution problems with infinite delay and an application to polymer rheology, SIAM J. Math. Anal. 15 (1984), 333 - 349.
- [22] M. Renardy, A local existence and uniqueness theorem for a K-BKZ fluid, Arch. Rational Mech. Anal. (to appear).
- [23] M. Renardy, Some remarks on the propagation and non-propagation of discontinuities in linearly viscoelastic liquids, Rheol. Acta 21 (1982), 251-254.
- [24] P. E. Rouse, A theory of the linear viscoelastic properties of dilute solutions of coiling polymers, J. Chem. Phys. 21 (1953), 1271-1280.
- [25] M. Slemrod, Instability of steady shearing flows in a nonlinear viscoelastic fluid, Arch. Rational Mech. Anal. 68 (1978), 211-225.
- [26] O. Staffans, On a nonlinear hyperbolic Volterra equation, SIAM J. Math. Anal. 11 (1980), 793-812.
- [27] W. Strauss, On continuity of functions with values in various Banach spaces, Pacific J. Math. 19 (1966), 543-551.

- [28] B. H. Zimm, Dynamics of polymer molecules in dilute solutions: viscoelasticity, flow birefringence and dielectric loss, J. Chem. Phys. 24 (1956), 269-278.

WJH/MR/jvs

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2667	2. GOVT ACCESSION NO. AD - A141700	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On a Class of Quasilinear Partial Integro-differential Equations With Singular Kernels		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) W. J. Hrusa and M. Renardy		8. CONTRACT OR GRANT NUMBER(s) MCS-8210950 & MCS-8215064 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS  See Item 18 below		12. REPORT DATE April 1984
		13. NUMBER OF PAGES 29
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office National Science Foundation P. O. Box 12211 Washington, DC 20550 Research Triangle Park North Carolina 27709		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Nonlinear viscoelasticity, hyperbolic equations, singular integrals, Volterra equations, energy estimates.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  We prove local and global existence theorems for a model equation in nonlinear viscoelasticity. In contrast to previous studies, we allow the memory function to have a singularity. We approximate the equation by equa- tions with regular kernels and use energy estimates to prove convergence of the approximate solutions.		